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The Tale of Sequences and Series

Sequences and series is the final BC Calculus topic. There are hundreds of sequences and series that happen around us every day. When we go to the bank and deposit money, the interest accumulates and adds to the original amount of money put in. That is actually a real- world example of a sequence. A sequence is essentially a list of numbers. A series would be adding the total amount of the money from each month together for a year. A series is like a sequence, because it’s a list of numbers. In a series, the terms are added together, unlike a sequence.

When a series converges, it means that the sum of the terms approaches a certain number thus resulting in the convergence of a series. When a series diverges, the sum of the terms just keep growing and adding up. The series continues to grow to infinity and does not approach a certain number, which results in the divergence of a series. A series can actually converge and diverge. A series can converge for a certain range of x-values and then diverge for the rest of the terms. To find the interval of convergence, a ratio test is conducted to find the x-values of where the series converges. From there, the end-points, the x-values found in the ratio test, are evaluated in the original function to determine if it is a closed interval, open interval, or closed on one end and open on the other. A sample problem is shown below to find the interval of convergence evaluated at the endpoints.

1. The series alternates ✓
2. ✓
3. ✓

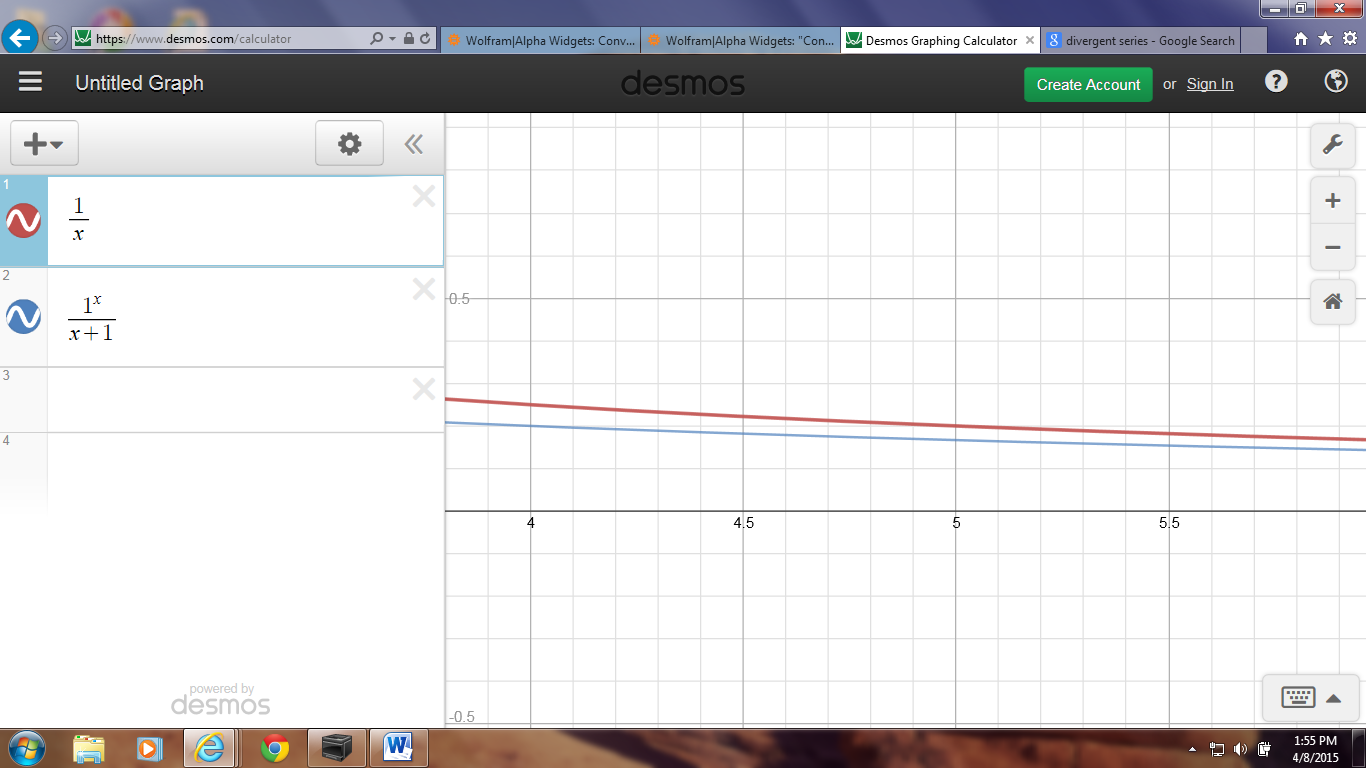


Figure 1. Direct Comparison Test (“Desmos Graphing Calculator”)

The series above converges from the interval of **.** The first endpoint evaluated was (-1/2), which converged by the alternating series test. The second endpoint evaluated was (1/2) which diverged by the Direct Comparison test. The series of diverged, and since the function of was below the divergent function, it also diverges.

Three sample series are shown below with steps and explanations about testing whether the series converges or diverges.

The limit comparison test was used above to find if the series converges of diverges. The limit comparison test was chosen because the ratio test was inconclusive and the other tests did not seem to work. The known diverging limit was 1/n and that was compared to the original series. L’Hospital’s Rule was used until the function could be evaluated. When evaluated, the limit is infinity. Since the limit is infinity and the known limit diverges, the original series diverges as well. After conducting the limit comparison test, the original series was found to diverge.

The series above converges by the alternating series test. The alternating series test was used because the (-1)n shows the series alternates. There are three parts to see if the series actually converges by the alternating series test. The first part is the series alternates, which is true. The next part is that the . This part is also true because one over infinity is essentially zero. The final part is that the n+1th term is less than the nth term. Since the n+1th would make the denominator larger, the entire fraction would then be smaller thus proving the final part to be true. By passing all three parts, the series converges by the alternating series test.

The series above diverges by the geometric series test. The geometric series test was used because there is a common ratio between all of the terms. The ratio is 4/3 which is greater than one. The geometric series test states that the series diverges if the ratio is greater than or equal to one. Since the ratio is greater than one, the series then diverges.

In a Taylor series, you find the series through to a specific term given in the problem. To find a Taylor series, you take the original function and keep taking the derivative up until the chosen term. If you are asked to go to the (x-a)3 term, you would continue to the third derivative. After each individual derivative, the given a-value is plugged into that derivative. The first term of the series is the a-value plugged into the original function. Each value found through the derivative is used as a constant in front of the (x-a). C will represent the constant found through the derivatives in the general form of the Taylor series shown below. F, the first term, will represent when the a-value is plugged into the original function which begins the Taylor series. The Taylor series is set equal to the original function.

The Taylor series form above is only shown up to the given example of the third derivative. The series keeps on going with the same pattern up until the given derivative where then it would just stop. Each constant is over a value factorial, and the (x-a) is also to that value power. The series starts off with zero being the first factorial, and power. Raised to the zero and zero factorial both equal one which is why the first term is just alone. From then on the values of the factorials and powers increase by one as shown above. The same value for each term will be in the factorial and in the exponent. An example of deriving the Taylor Series of tanx through the (x-a)3 term is shown below with steps.

A Maclaurin series is similar to the Taylor series because the derivative is taken to the certain term. The factorial and power values are also the same in each term. A Maclaurin series is essentially a Taylor series just expanded about x=0. Instead of having (x-a) as the constant term, the term is just x, because the a-value would be zero. To find the first term, zero is plugged in to the original function and evaluated. The basic form of the Taylor series is shown below.

The example above is just shown up to the third derivative. The c is the value when zero is plugged into that derivative. F is the first term, and this is all set equal to the original given function. When used to approximate a certain value, like, the process is very easy. The series would then be set equal to that value and 1.1, without the square root, would be plugged in for x. The series would then be evaluated when 1.1 is plugged in. As the series keeps on adding up terms, it gets closer to the true value of . An example of deriving the Maclaurin Series of (1-x)-2 showing at least four terms is shown below with steps.

The Taylor series of (x+1)n is shown below with steps demonstrating how to find the interval of convergence.

The Taylor Series above converges on the interval of -2<x<0. When evaluated at the endpoints, the endpoints both diverged by the nth term test. Since the limit as n approaches infinity of both series does not equal zero, both of the series then diverge at the endpoints resulting in open ended intervals.

Since the Taylor Series about x=-1 is geometric, the sum of the series on the interval of convergence can be found. The form of the sum is shown below.

The sum is equal to a, the first term, over 1-r, the ratio. The common ratio in the Taylor series is x+1. The plugged in values to find the sum is shown below.

Thus, the sum is 1/-x on the interval of convergence found in the first part which is -2<x<0.

The intermediate value theorem can be used to find g(-0.5) by the function of g being defined as By the intermediate value theorem, g(x) must be equal to the integral of f(x)dx from -1 to -0.5. To find g(-0.5) the steps and solution is shown below.

The next part asks to let h be the function defined by h(x)=f(x2-1). It then proceeds to ask to find the first three nonzero terms and the general term of the Taylor series for h about x=0. Finally, it asks to find the value of h(0.5). The series is shown below where n stands for the term.

H(0.5) can be found by using the steps below and the information that the sum of f(x) is 1/-x found in part two.

There are three simple ways to compute error. The first way to compute error is actually computing the error. In order to actually compute error, you take the actual value and subtract the estimated value from it. Actually computing error seems to be the easiest way to find error. The next way to find error is the Lagrange error. Lagrange error is used in Taylor series and Maclaurin series. The Lagrange method finds the remainder to estimate the accuracy of using the partial sum. To find the remainder, the form is shown below.

The absolute value of R represents the remainder. M is the maximum value or upper bound of the series. The n+1 is the partial sum value you are going to. So if it asks for the 11th partial sum, 11 would go in both places of n+1. The absolute value of x-a is the given value of x minus the value of a or what the series is about. The amount of zeros after the decimal place shows the accuracy. If there are three zeros, the estimate is accurate to the third decimal place. The Alternating Series error approximation is the final way to find error. First, the given series has to pass the Alternating Series Test which states that the series alternates, the limit of the series as n approaches infinity equals zero, and that the n+1th term is less than the nth term. The format for the Alternating Series error approximation is shown below.

The format above states that the error, Rn, will be no larger than the value found in the given partial sum, tn, plus one term. So if we are finding the 3rd partial sum, then the error will be less than the fourth term evaluated with the given x value.

An example of the Lagrange method for finding error is shown below with work and explanations.

Actual value: 1.1051709180756

Estimated Value: 1.1051708333334

Error: 0.0000000847422

In the sample above, the value and error of e0.1 was to be found using partial sum. Lagrange’s method was used to find this. The first step shows the 5th partial sum written out of just ex. The next step shows the 5th partial sum of e0.1 and that estimated value. The third step shows the remainder which finds the accuracy. This estimation is accurate to the 7th decimal place according to the Lagrange method. The error between the actual and estimated value demonstrate the Lagrange’s method of estimating accuracy is true, because the estimate is accurate to seven decimal places according to the found actually error.

In conclusion, sequences and series are not that difficult of topics to understand. Sequences and series are closely related. Just as a refresher, sequences are basically lists of numbers and series is adding those numbers up. Just to sum things up, series can converge, approach a number, or diverge, continue on to infinity. Some series can converge for a certain period and then diverge for the rest. Another subtopic is that there are two types of series that can be derived from certain values. Those series are the Taylor and Maclaurin series. The final subtopic covered was error. Again, there are three types of error: Lagrange, Alternating Series, and actual error. Overall, sequences and series is a topic filled with many little things that make up the entire topic as a whole. Sequences and series isn’t that difficult to understand, but constant practice and application of the practice always helps students grasp the topic better.

Works Cited

"Desmos Graphing Calculator." *Desmos Graphing Calculator*. N.p., n.d. Web. 9 Apr. 2015. <https://www.desmos.com/calculator>.