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Advanced Placement Calculus

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Summing Things Up

In society, we always run towards our calculator to help solve our problems, well, our math problems at least. For a good portion of the time, we use our calculator to find the area under a curve or graph. A lot may not know that we can actually calculate these values by hand. Although calculation by hand is a very time consuming aspect, it could come in handy is detrimental situations. When you are face to face with a killer and they ask you to compute the area under a curve by hand to save your life, you will know how to because in this paper I am going to explain three different ways that you can answer this problem by hand. The three different ways to compute the area under the function are through Riemann Sums, The Trapezoid Rule, and Simpsons Rule.

A Riemann sum is a sum where each term represents an area of a triangle with altitude f(x) and a base of dx. These areas are all added up to find the area under the curve. There are five different types of rectangles used for Riemann sums. The five different types are left, midpoint, right, lower, and upper.

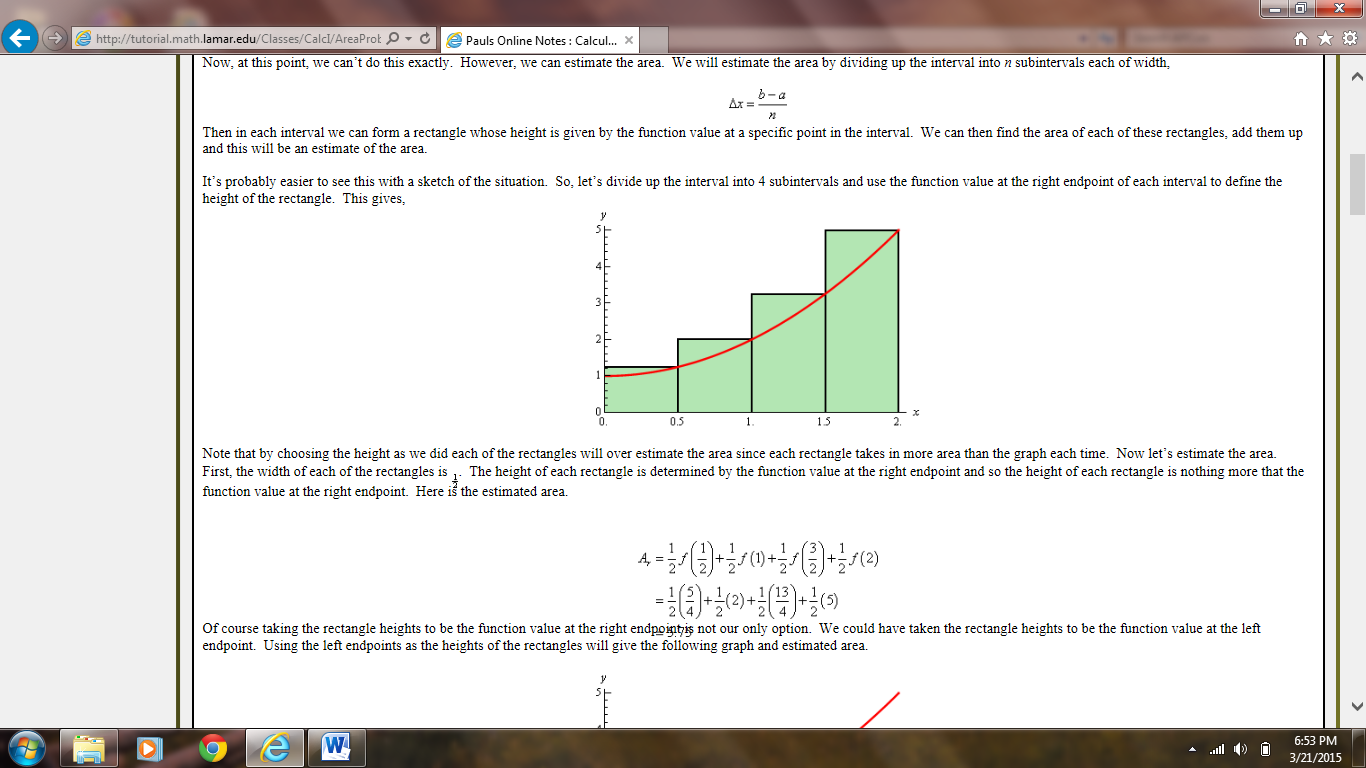
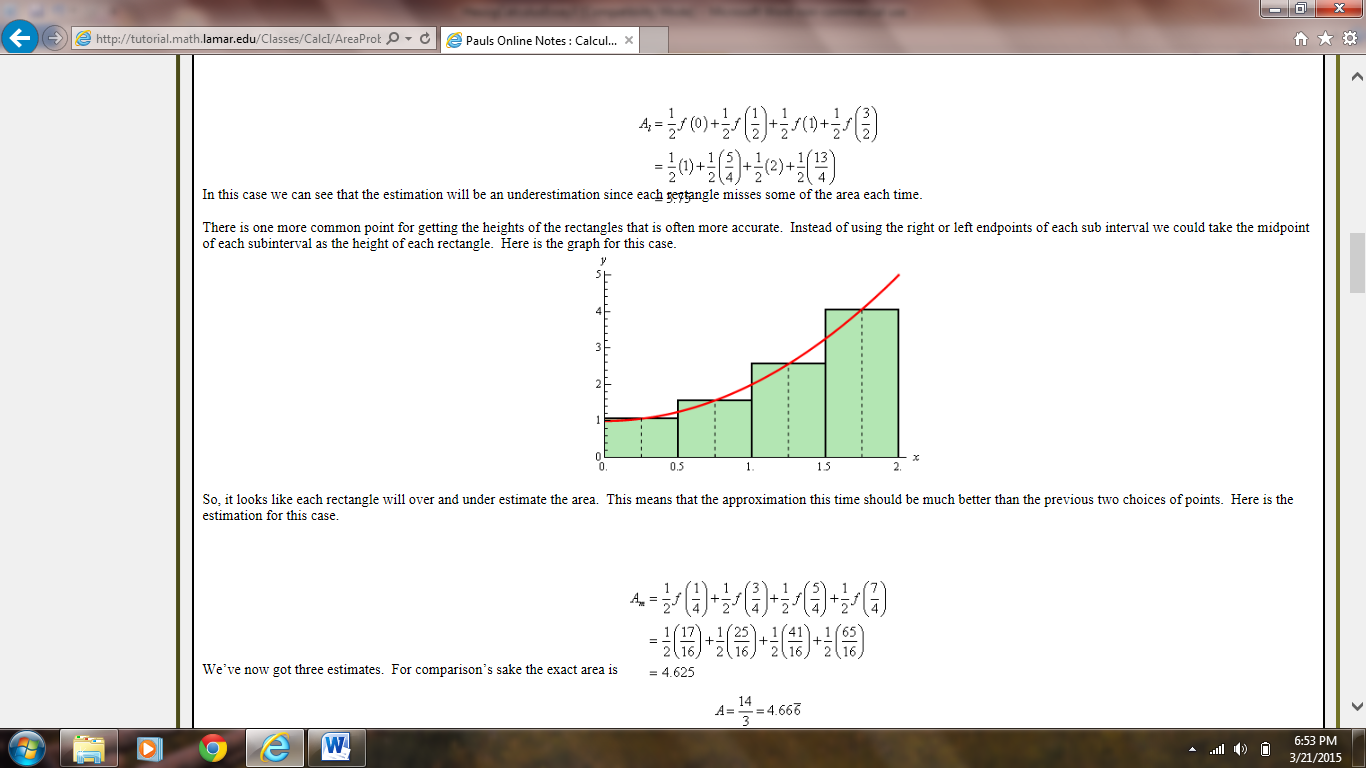
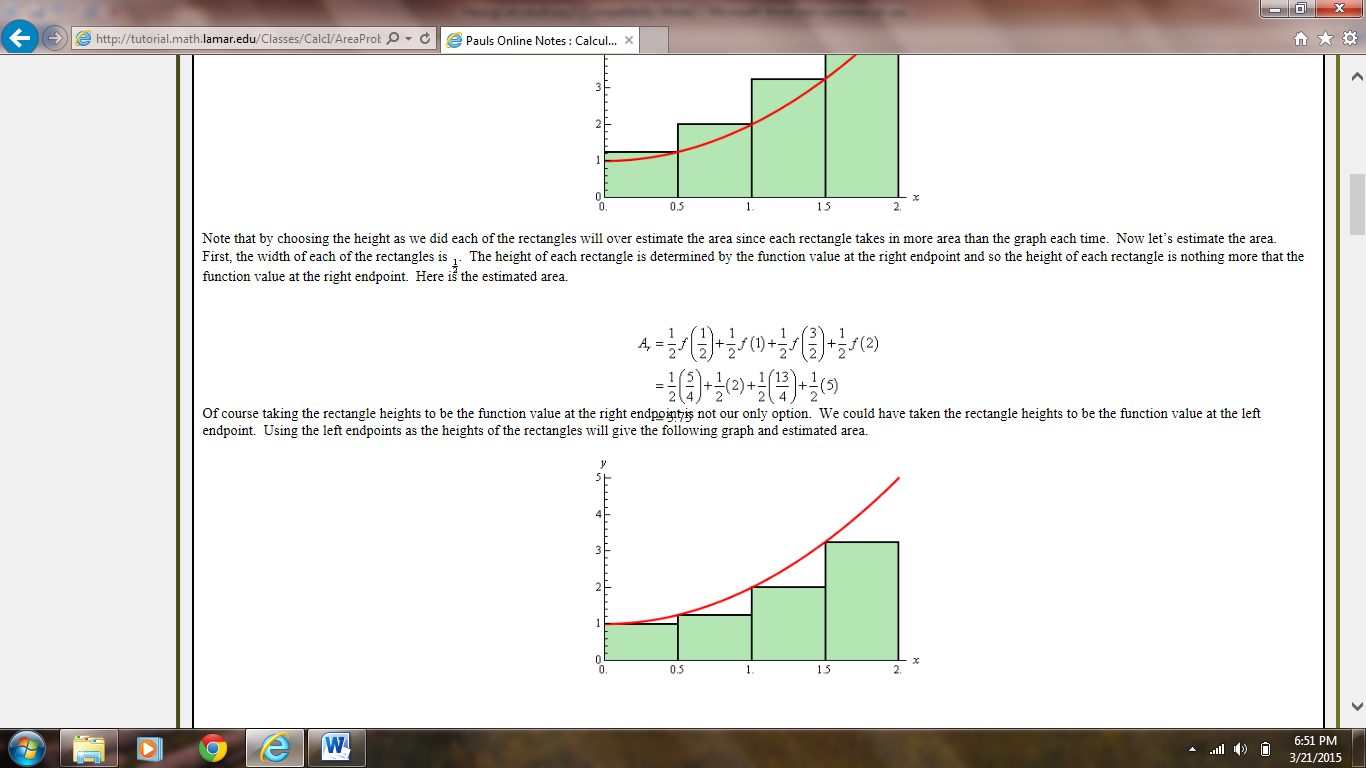


Figure 1. Riemann Sum Triangles (“Pauls Online Notes: Calculus l”)

Figure 1 shows the left, midpoint, and right Riemann sums. Each rectangle has a different part touching the graph. For the left rectangles, the rectangle hits the leftmost point of the curve. For the right rectangles, the rectangle hits the rightmost point of the curve. For the midpoint rectangles, the rectangle hits the middle point of each curve. To find the total area under the curve, you add up the area of each individual rectangle which is found by multiplying the altitude, f(x), and the base, dx.

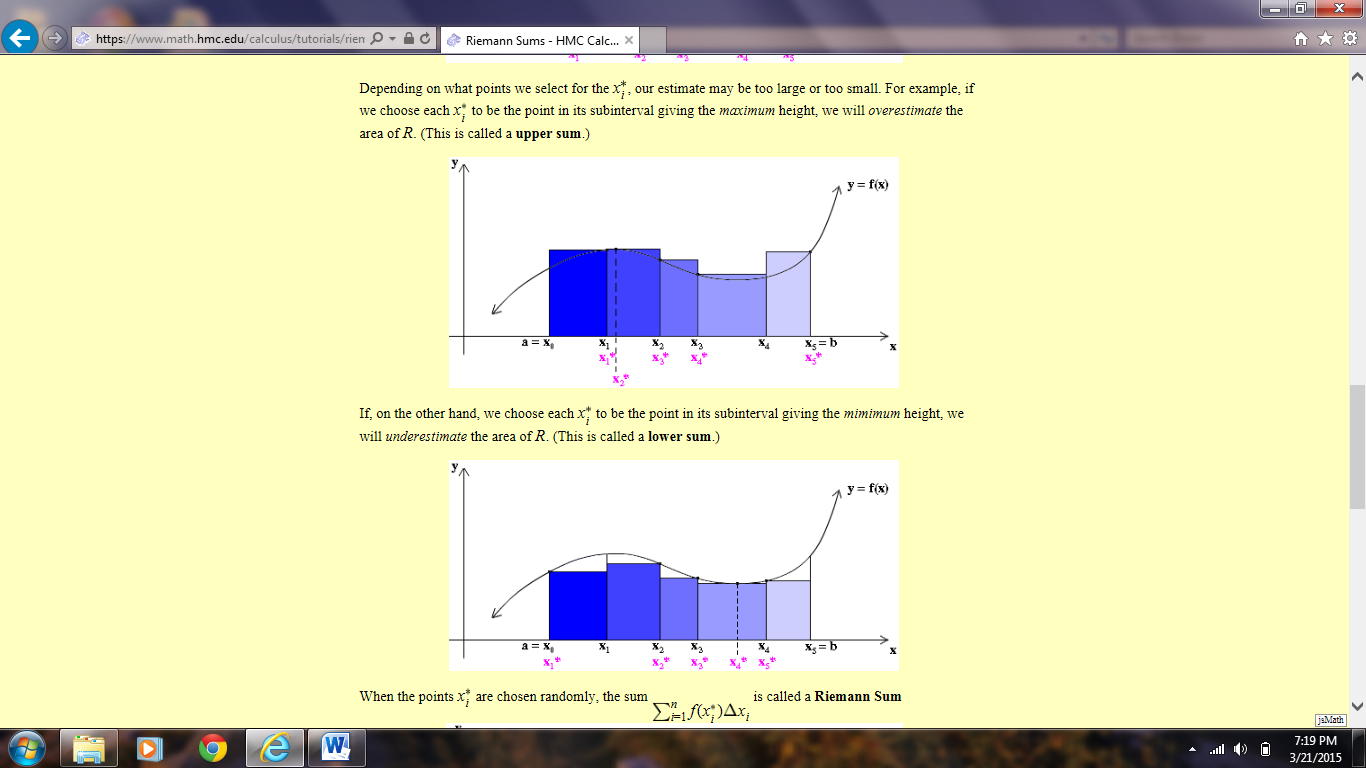
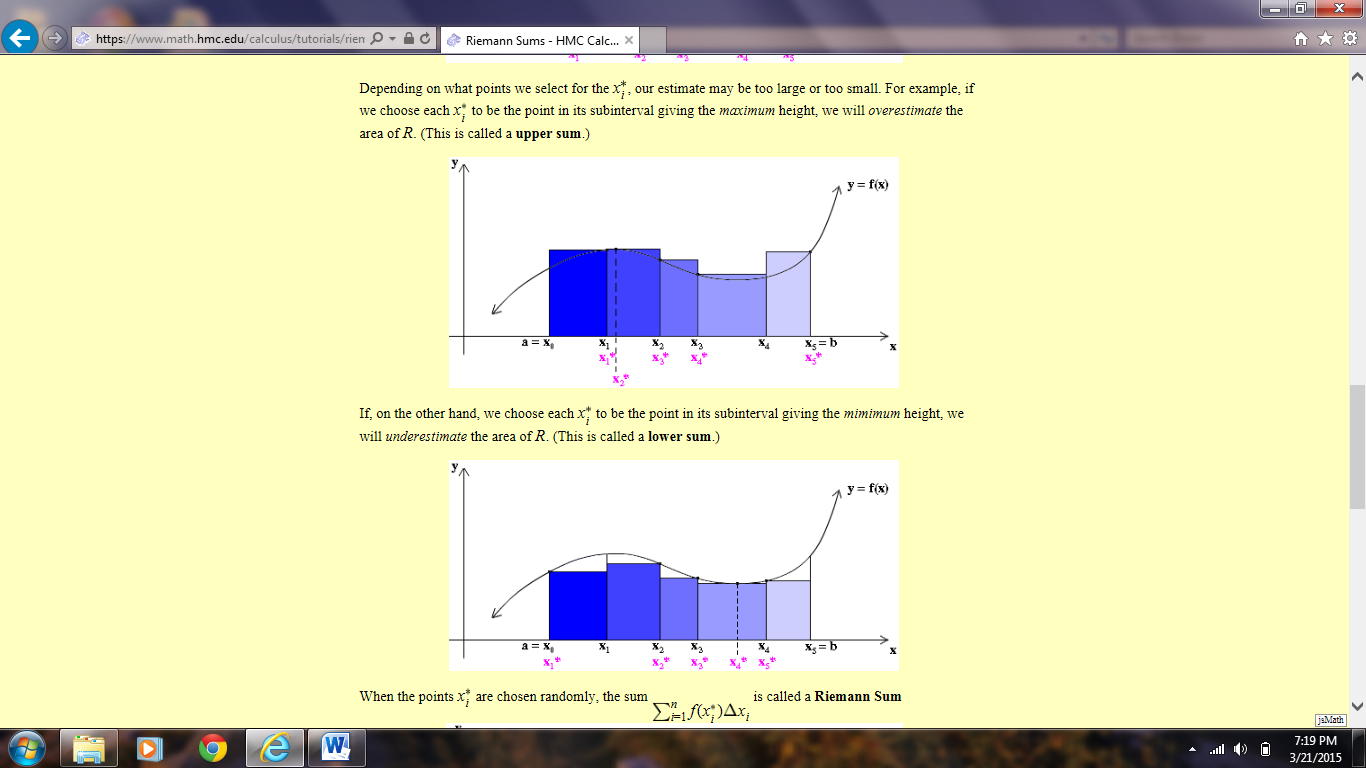
 

Figure 2. Lower and Upper Riemann Sums (“Riemann Sums”)

Figure 2 shows the lower and upper Riemann sums. Each rectangle has a different part touching the graph. For the lower rectangles, the rectangle hits the lowest point of the curve. For the upper rectangles, the rectangle hits the upmost point of the curve. To find the total area under the curve, you add up the area of each individual rectangle which is found by multiplying the altitude, f(x), and the base, dx.

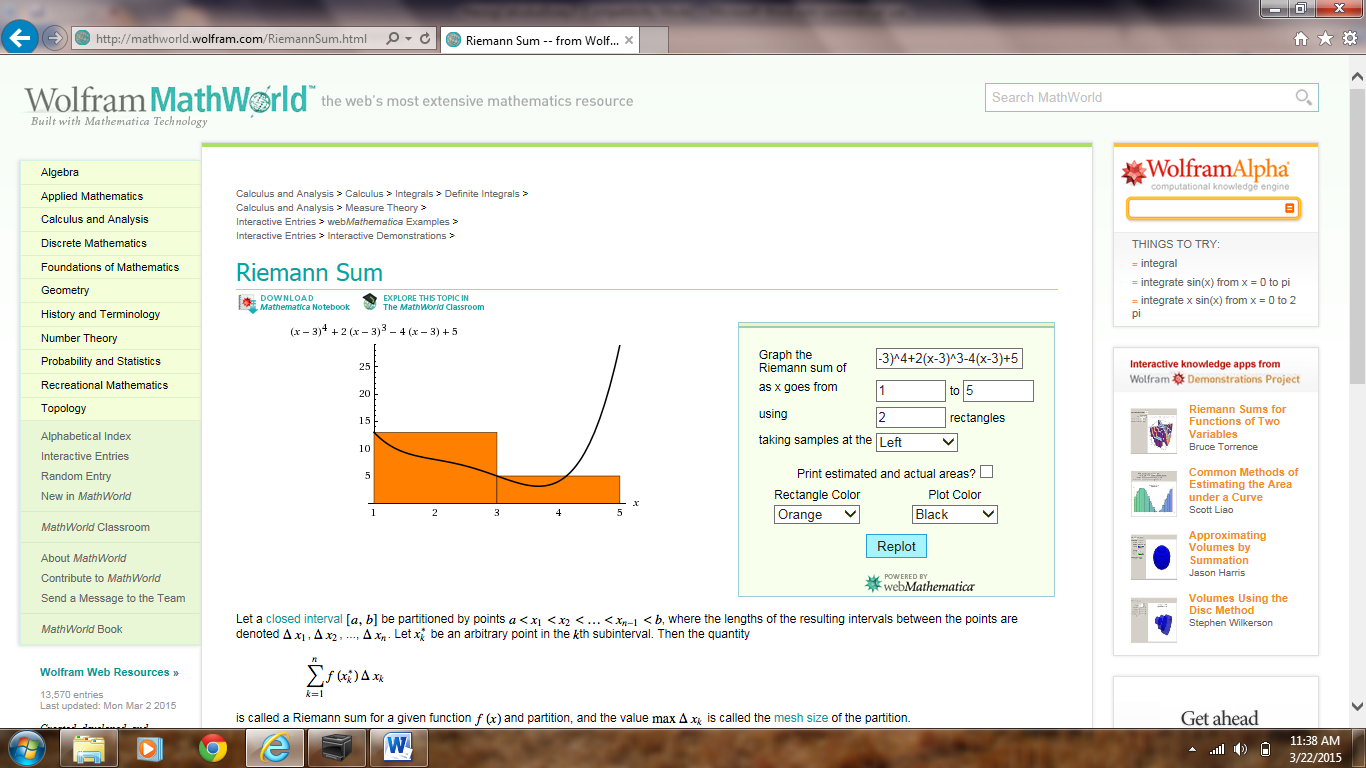


Figure 3. Left Riemann Sum (“Riemann Sum”)

Figure 3 shows the left Riemann sum of the given equation shown below.

Left Riemann sums are placed at the left most points of the intervals. The graph above was evaluated from x=1 to x=5. The graph was split into two intervals. The first interval, or rectangle, is from 1 to 3. The next interval is from 3 to 5. The left most point for the first rectangle is 1, so the rectangle is drawn based off of that point. The left most point for the next rectangle is 3, so the rectangle is drawn based off of that point. The f(x) value for the left most point represents the altitude of the entire rectangle. The size of the interval represents the base of the rectangle. To find the total area under the curve, you add up the area of each individual rectangle which is found by multiplying the altitude, f(x), and the base, dx. A sample calculation of the rectangles above is shown below.

The approximation of the left Riemann sum is more than the definite integral where the true area under the curve is 32 units2.

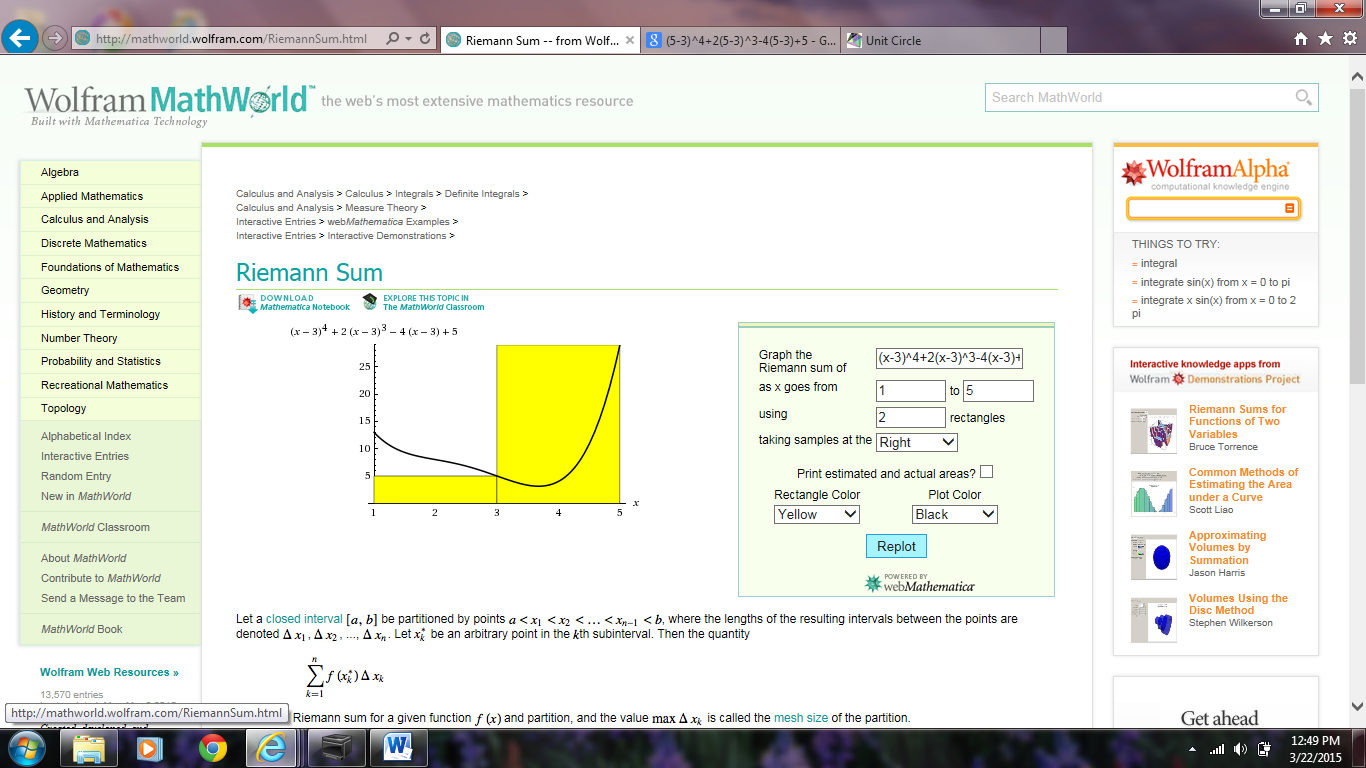


Figure 4. Right Riemann Sum (“Riemann Sum”)

Figure 4 shows the right Riemann sum of the given equation shown below.

Right Riemann sums are placed at the right most points of the intervals. The graph above was evaluated from x=1 to x=5. The graph was split into two intervals. The first interval, or rectangle, is from 1 to 3. The next interval is from 3 to 5. The right most point for the first rectangle is 3, so the rectangle is drawn based off of that point. The right most point for the next rectangle is 5, so the rectangle is drawn based off of that point. The f(x) value for the right most point represents the altitude of the entire rectangle. The size of the interval represents the base of the rectangle. To find the total area under the curve, you add up the area of each individual rectangle which is found by multiplying the altitude, f(x), and the base, dx. A sample calculation of the rectangles above is shown below.

The approximation of the right Riemann sum is more than the definite integral where the true area under the curve is 32 units2.



Figure 5. Midpoint Riemann Sum (“Riemann Sum”)

Figure 5 shows the midpoint Riemann sum of the given equation shown below.

Midpoint Riemann sums are placed at the middle point of each interval. The graph above was evaluated from x=1 to x=5. The graph was split into two intervals. The first interval, or rectangle, is from 1 to 3. The next interval is from 3 to 5. The midpoint for the first rectangle is 2, so the rectangle is drawn based off of that point. The midpoint for the next rectangle is 4, so the rectangle is drawn based off of that point. The f(x) value for the midpoint represents the altitude of the entire rectangle. The size of the interval represents the base of the rectangle. To find the total area under the curve, you add up the area of each individual rectangle which is found by multiplying the altitude, f(x), and the base, dx. A sample calculation of the rectangles above is shown below.

The approximation of the midpoint Riemann sum is more than the definite integral where the true area under the curve is 32 units2.

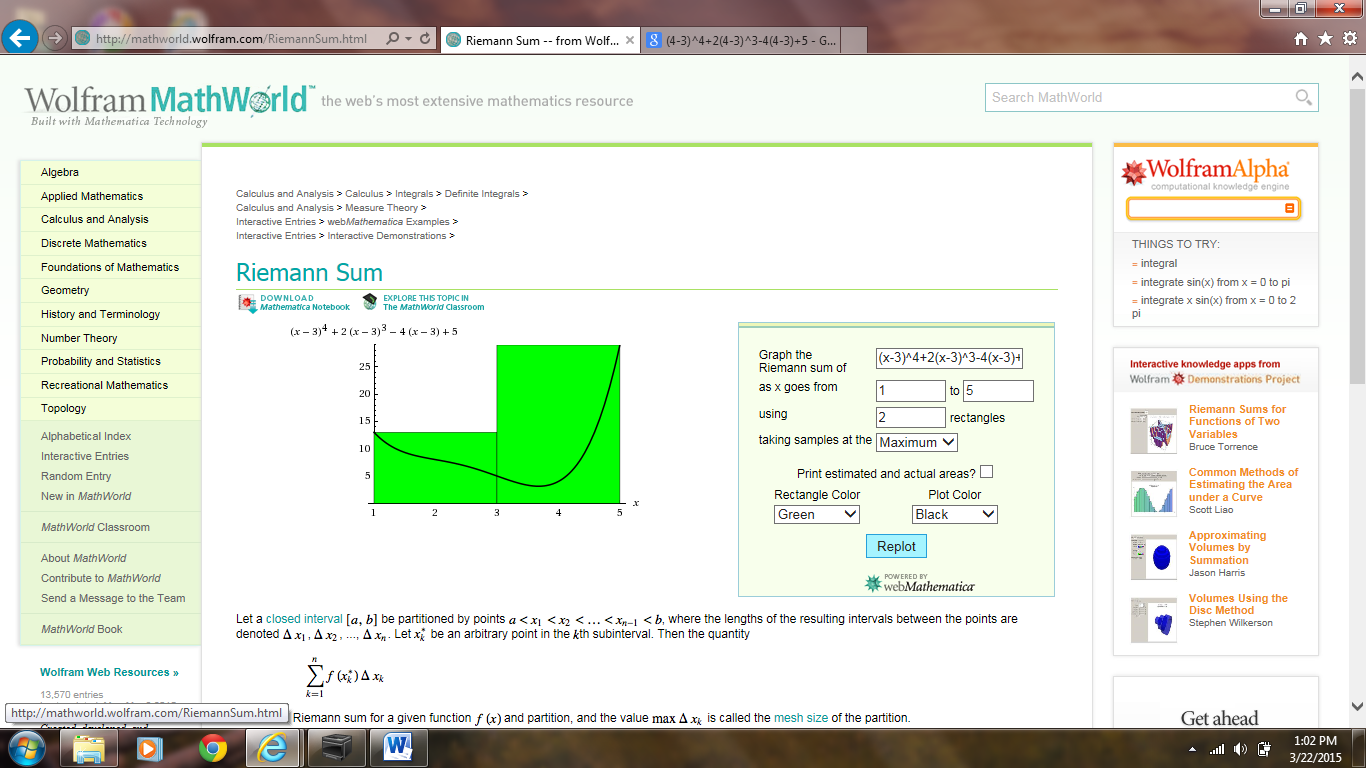


Figure 6. Upper Riemann Sum (“Riemann Sum”)

Figure 6 shows the upper Riemann sum of the given equation shown below.

Upper Riemann sums are placed at the upmost points of the intervals. The graph above was evaluated from x=1 to x=5. The graph was split into two intervals. The first interval, or rectangle, is from 1 to 3. The next interval is from 3 to 5. The upmost point for the first rectangle is 1, so the rectangle is drawn based off of that point. The upmost point for the next rectangle is 5, so the rectangle is drawn based off of that point. The f(x) value for the upmost point represents the altitude of the entire rectangle. The size of the interval represents the base of the rectangle. To find the total area under the curve, you add up the area of each individual rectangle which is found by multiplying the altitude, f(x), and the base, dx. A sample calculation of the rectangles above is shown below.

The approximation of the upper Riemann sum is more than the definite integral where the true area under the curve is 32 units2.

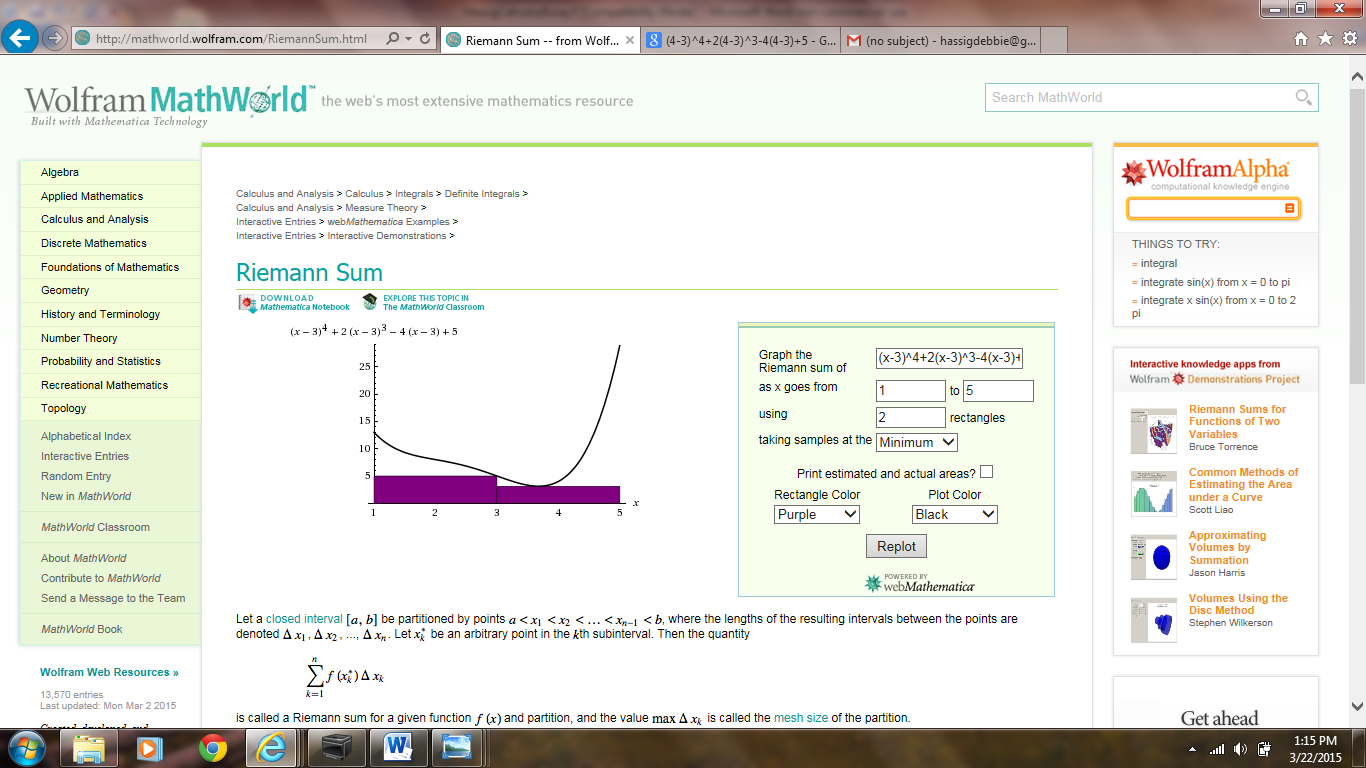


Figure 7. Lower Riemann Sum (“Riemann Sum”)

Figure 7 shows the lower Riemann sum of the given equation shown below.

Lower Riemann sums are placed at the lowest points of the intervals. The graph above was evaluated from x=1 to x=5. The graph was split into two intervals. The first interval, or rectangle, is from 1 to 3. The next interval is from 3 to 5. The lowest point for the first rectangle is 3, so the rectangle is drawn based off of that point. The lowest point for the next rectangle is 3.67765, so the rectangle is drawn based off of that point. The f(x) value for the lowest point represents the altitude of the entire rectangle. The size of the interval represents the base of the rectangle. To find the total area under the curve, you add up the area of each individual rectangle which is found by multiplying the altitude, f(x), and the base, dx. A sample calculation of the rectangles above is shown below.

The approximation of the lower Riemann sum is less than the definite integral where the true area under the curve is 32 units2.

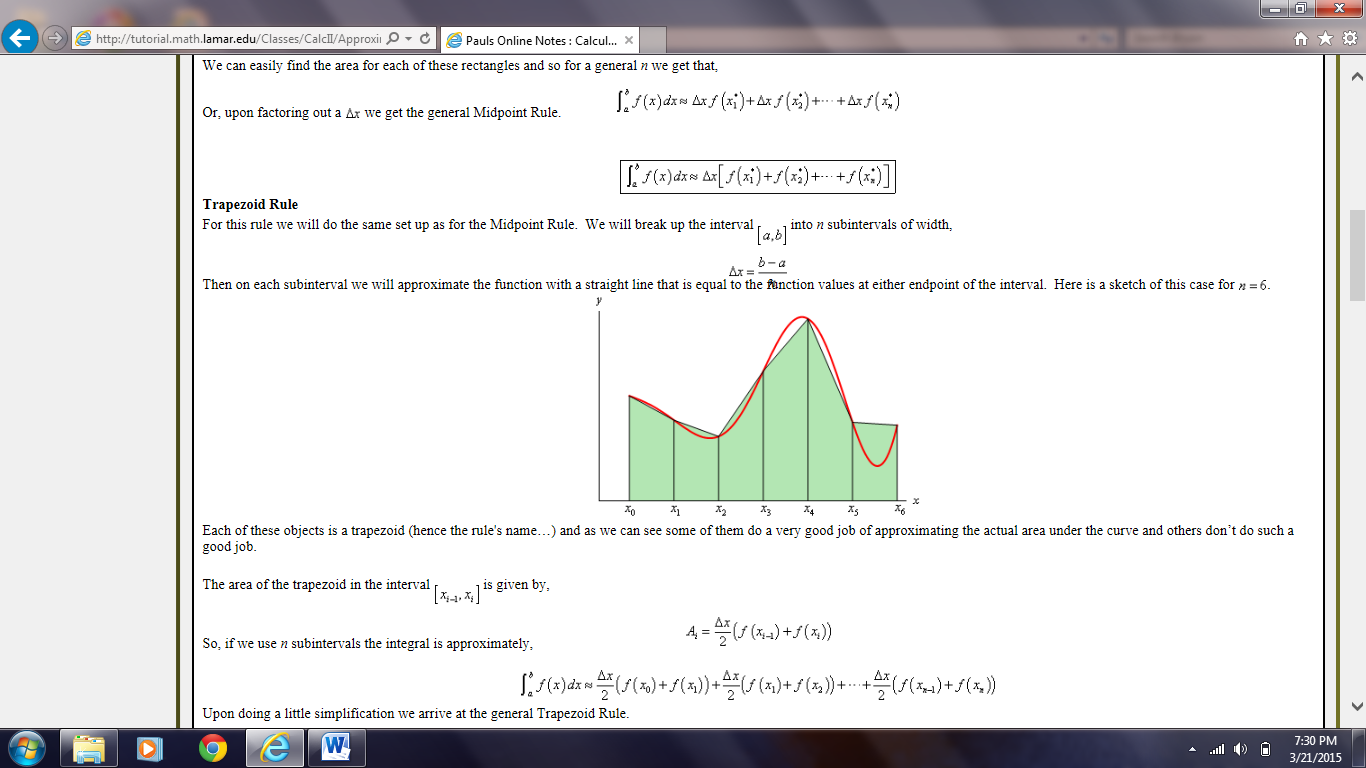


Figure 8. Trapezoid Rule (“Pauls Online Notes: Calculus ll”)

The Trapezoid Rule is similar to Riemann sums, except trapezoids are used instead of rectangles. The trapezoid is formed along the curve as shown above in Figure 8. Each trapezoid is formed with a width of dx and two altitudes, f(x), which is the height of the sides of the trapezoid. To find the area of each trapezoid, you add the two heights up, multiply it by a half, and then multiply that by dx, or the width of the trapezoid. The equation to find the total area of all the trapezoids is shown below.

To find the area, you add up each height, but after the first height, each additional height is multiplied by two until the last height is reached where it is just that single height. Then that sum of the heights is multiplied by a half which is then multiplied by dx, the width of each trapezoid.

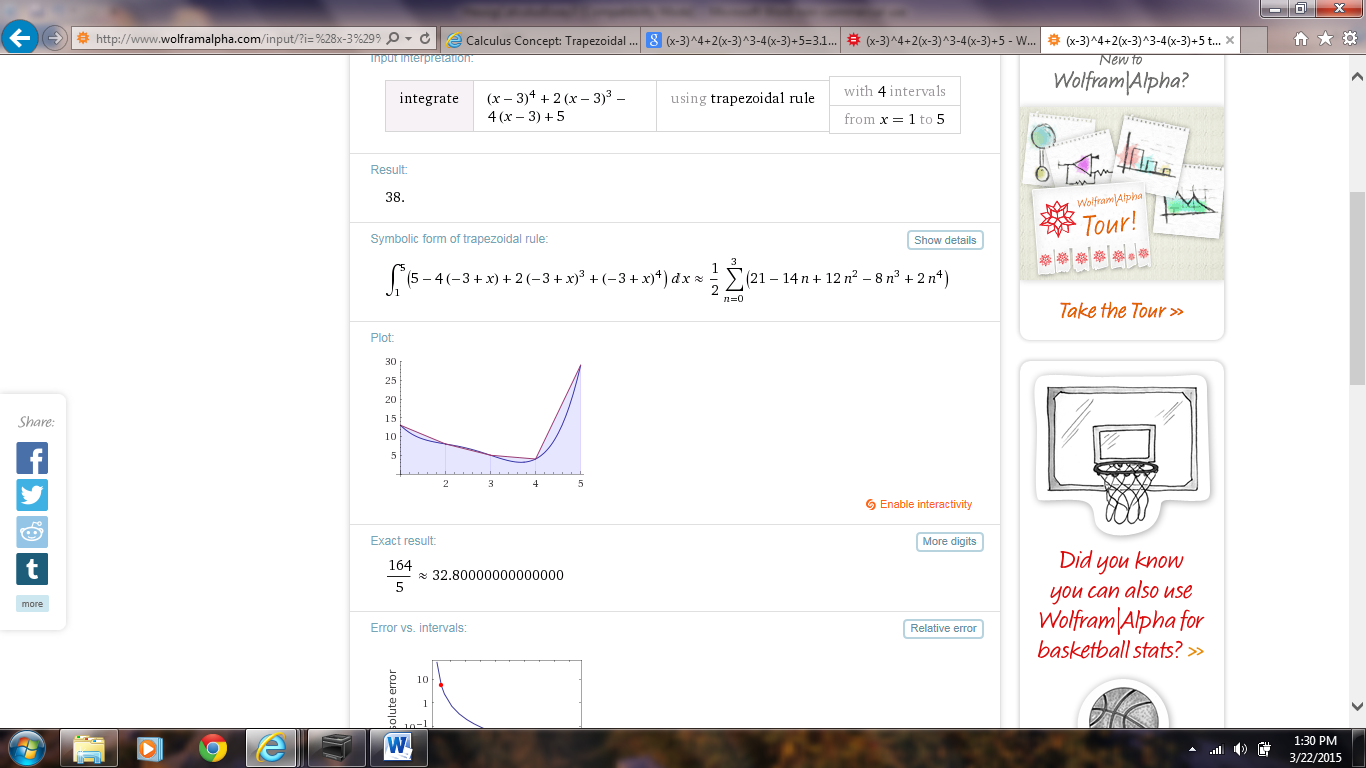


Figure 9. Trapezoid Rule (“Wolfram Alpha”)

The trapezoid is formed along the graph as shown above in Figure 9. The equation of the graph is shown below.

The graph is separated into four trapezoids from the interval of x=1 to x=5. Each trapezoid is formed with a width of dx, in this case the width is 1, and two altitudes, f(x), which is the height of the sides of the trapezoid. To find the area of each trapezoid, you add the two heights up, multiply it by a half, and then multiply that by dx, or the width of the trapezoid. A sample equation of the graph above is shown below.

The approximation of the trapezoid rule is more than the definite integral where the true area under the curve is 32 units2.

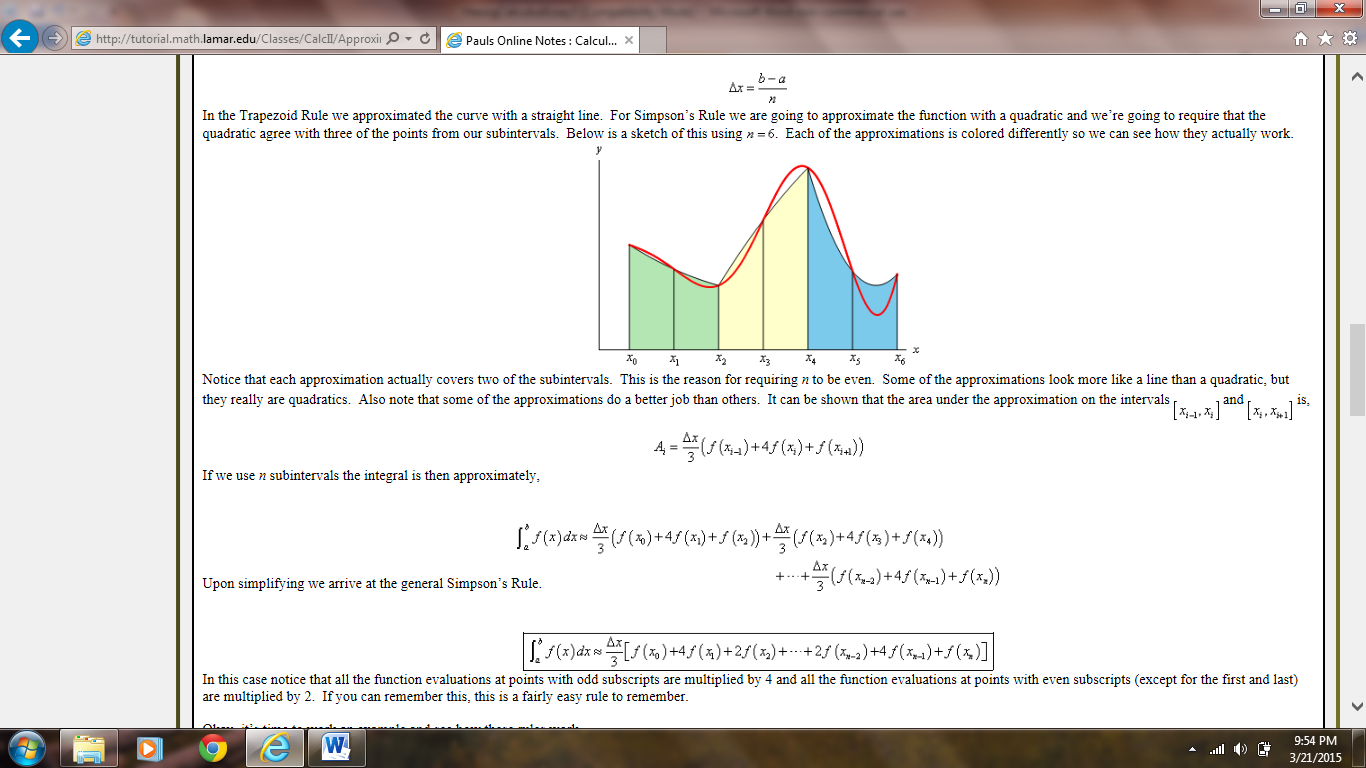


Figure 10. Simpson’s Rule (“Pauls Online Notes: Calculus ll”)

Simpson’s Rule is similar to the Trapezoid Rule and Riemann Sums except Simpson’s Rule is the most accurate because instead of using rectangles and trapezoids, parabolas are formed to mimic the graph. As shown above in Figure 10, the colored sections follow a parabolic shape similar to the original graph. To find the area under the curve, Simpson’s Rule has a specific formula shown below.

In the formula above, the heights are the beginning and end y coordinates. The y coordinates starts off being multiplied by one, then the next coordinates are multiplied by alternating fours and twos, but the alternating sequence always starts with a 4. The last y coordinate is then multiplied by one, just like the first. The entire sum of the heights is multiplied by one-third and that quantity is then multiplied by h which is the change in x.

The Mean Value Theorem for Integrals states that if a function, f, is continuous on [a,b] such that . Essentially, the area under the graph will have the same area as the rectangle formed with height f(c) at x=c. A sample calculation is shown below for the given f(x) equation.

on [1,5]

The f(c) value found above is the altitude of the rectangles. The c values also found above are the x-points that coordinate with the altitude. Each c-value is in the closed interval of 1 to 5, so both values are used. Since the graph is split into two intervals, one point falls into each interval. The Mean Value Theorem graph is shown in Figure 11.



Figure 11. Mean Value Theorem for Integrals Graph (“Desmos Graphing Calculator”)

The horizontal line in Figure 11 lies at the value of y=8.2 which is the altitude, f(c), of both rectangles. The vertical lines that lie in each of the two intervals are the c values that correspond to the f(c) value. In the first interval or rectangle, the c value is at x=1.901. In the second interval, the c value is at x=4.371. The Mean Value Theorem for Integrals states that the area of the shaded rectangles is equal to the area under the curve from the closed interval of x=1 to x=5.

Table 1

Balloon Problem

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| t (seconds) | 0 | 1 | 4 | 7 | 11 | 12 |
| r’(t) (ft/sec) | 5.7 | 4.0 | 2.0 | 1.4 | 0.5 | 0.4 |

Computing the radius of a balloon when t=7.2 using a tangent line approximation at t=7 is very simple. According to the table above, the rate of change or derivative is the slope of the tangent line. So when t=7, the slope is 1.4 feet/second. There is given information that when t=7, the radius is 32 feet. All of this information can be plugged into point slope form to get the following equation for the tangent line shown below.

The radius when t=7.2 is 32.28 feet which is an over approximation of the true value. This is over approximation because at 7 seconds the radius is 32, so the difference is too large to correspond with the rate of change of the balloons radius.

Finding the rate of change of the volume of a balloon at t=7 is also a simple process. A sample calculation and explanation is shown below.

The volume equation’s derivative is taken in order to find the rate of change of the volume. Next, the known values for when t=7 are plugged in. At t=7 the radius is 32 feet and the rate of change of the radius is 1.4 feet/second. When all those values are plugged in, the rate of change of the volume of the balloon is 18,015.1 ft/sec3.

Figure 12. Right Riemann Sum

Figure 12 shows a right Riemann sum. Right Riemann sums are placed at the right most points of the intervals. The graph above was evaluated from x=0 to x=12. The graph was split into five intervals. The first interval, or rectangle, is from 0 to 1. The next interval is from 1 to 4. The third interval is from 4 to 7. The fourth interval is from 7 to 11. The final interval is from 11 to 12. The right most point for the first rectangle is 1, so the rectangle is drawn based off of that point. The right most point for the next rectangle is 4, so the rectangle is drawn based off of that point. The right most point for the third rectangle is 7, so the rectangle is drawn based off of that point. The right most point for the fourth rectangle is 11, so the rectangle is drawn based off of that point. The right most point for the final rectangle is 12, so the rectangle is drawn based off of that point. The f(x) value for the right most point represents the altitude of the entire rectangle. The f(x) values for the rectangles are shown on the graph and are shown in Table 1. The size of the interval represents the base of the rectangle. In this case, each rectangle has a different base. To find the total area under the curve, you add up the area of each individual rectangle which is found by multiplying the altitude, f(x), and the base, dx. A sample calculation of the rectangle above is shown below.

The area found above is related to the definite integral shown below.

The definite integral shown above is the sum of the rate at which the radius of the balloon is changing adds up to the radius of the balloon at a specific time which is from 0 to 12 seconds. The approximation of the right Riemann sum is less than the definite integral because the rectangles all lie completely below the line so there is open space between the curve and the rectangles that is unaccounted for and not made up anywhere else.

Overall, most math concepts are not that hard to understand. With lots of practice and hard work, difficult topics can be made easy. Riemann sums, the Trapezoid Rule, and Simpsons Rule all allow you to find the area under the curve by hand. Although these ways are not always exact, Simpsons Rule is the most accurate. These topics will help the students understanding of the area under the curve by allowing them to visually see everything add up. Summing things up is just another part to understanding calculus topics as a whole.

Works Cited

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